NOTES ON CONVERGENCE OF THE FOURIER SERIES

PAWEŁ JAN PISKORZ

Abstract. Our notes contain a way to prove the point convergence of any Fourier series of a piecewise periodic continuous smooth function. The key technique we use is the computation of the limit of the partial sum of the Fourier series. To achieve that result we apply integration with substitution of variables and the Dirichlet integral value.

1. Introduction

Let us consider partial sum $s_N(x)$ of the Fourier series

$$s_N(x) = \frac{a_0}{2} + \sum_{k=1}^{N}(a_k \cos kx + b_k \sin kx) \tag{1.1}$$

where

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx \tag{1.2}$$

for $k = 0, 1, 2, \dots, N$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx \tag{1.3}$$

for $k = 1, 2, \dots, N$.

Substituting the expressions for the Fourier coefficients we obtain

$$\begin{aligned}
s_N(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt + \frac{1}{\pi}\sum_{k=1}^{N}\Big[\int_{-\pi}^{\pi} f(t)\cos kt\,dt\cdot\cos kx + \\
&\qquad + \int_{-\pi}^{\pi} f(t)\sin kt\,dt\cdot\sin kx\Big] \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\Big[\frac{1}{2} + \sum_{k=1}^{N}(\cos kt\cos kx + \sin kt\sin kx)\Big]\,dt = \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\Big[\frac{1}{2} + \sum_{k=1}^{N}(\cos k(t-x))\Big]\,dt
\end{aligned} \tag{1.4}$$

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2. Sum of cosines formula derivation

We use a formula for the sum of the geometric series $\sum_{k=1}^{N} e^{ik\varphi}$ as in [Hir] and for convenience presented with modifications in Section 5.

$$\sum_{k=1}^{N} e^{ik\varphi} = \sum_{k=1}^{N} (\cos k\varphi + i \sin k\varphi) \tag{2.1}$$
$$= \cos((N+1)\varphi/2)\frac{\sin N\varphi/2}{\sin\varphi/2} + i\sin((N+1)\varphi/2)\frac{\sin N\varphi/2}{\sin\varphi/2}$$

where $i^2 = -1$. Computing from equations

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \tag{2.2}$$

and

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \tag{2.3}$$

we obtain

$$\cos\alpha\sin\beta = \frac{1}{2}(\sin(\alpha+\beta) - \sin(\alpha-\beta)) \tag{2.4}$$

$$\sin\alpha\sin\beta = -\frac{1}{2}(\cos(\alpha+\beta) - \cos(\alpha-\beta)) \tag{2.5}$$

and we can substitute

$$\alpha = (N+1)\varphi/2 \qquad \beta = N\varphi/2 \tag{2.6}$$

$$\alpha + \beta = N\varphi + \varphi/2 \qquad \alpha - \beta = \varphi/2 \tag{2.7}$$

After computations we arrive at

$$\sum_{k=1}^{N} \sin k\varphi = -\frac{1}{2}(\cos(N\varphi + \varphi/2) - \cos\varphi/2)\frac{1}{\sin\varphi/2} \tag{2.8}$$

$$\sum_{k=1}^{N} \cos k\varphi = \frac{1}{2}(\sin(N\varphi + \varphi/2) - \sin\varphi/2)\frac{1}{\sin\varphi/2} \tag{2.9}$$

From equation (2.9) we receive

$$\frac{1}{2} + \sum_{k=1}^{N} \cos k\varphi = \frac{\sin(N\varphi + \varphi/2)}{2\sin\varphi/2} \tag{2.10}$$

3. Applying the sum of cosines formula

We apply the formula from equation (2.10) for the sum of cosines having $\varphi = t - x$ in equation (1.4)

$$s_N(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,\frac{\sin(N(t-x)+(t-x)/2)}{2\sin(t-x)/2}\,dt \tag{3.1}$$

We substitute $t - x = u$. Then we have the result obtained in [Tol]

$$\begin{aligned} s_N(x) &= \frac{1}{\pi}\int_{-\pi-x}^{\pi-x} f(x+u)\,\frac{\sin(Nu+u/2)}{2\sin u/2}\,du = \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+u)\,\frac{\sin(Nu+u/2)}{2\sin u/2}\,du \end{aligned} \tag{3.2}$$

Now we substitute in equation (3.2) $Nu = \alpha$. Then we have $u = \alpha/N$, $u/2 = \alpha/2N$, $du = d\alpha/N$. For the limits of integration if $u = -\pi$ then $\alpha = -N\pi$ and if $u = \pi$ then $\alpha = N\pi$. In this way we obtain

$$\begin{aligned} s_N(x) &= \frac{1}{\pi}\int_{-N\pi}^{N\pi} f(x+\alpha/N)\frac{\sin(\alpha+\alpha/2N)}{2N\sin\alpha/2N}\,d\alpha = \\ &= \frac{1}{\pi}\int_{-N\pi}^{N\pi} f(x+\alpha/N)\frac{\sin(\alpha+\alpha/2N)}{\alpha(\sin\alpha/2N)/(\alpha/2N)}\,d\alpha \end{aligned} \tag{3.3}$$

4. Taking the limit and the final result

In particular for $N \to \infty$

$$\begin{aligned} &\lim_{N\to\infty} s_N(x) = \\ &= \lim_{N\to\infty}\frac{1}{\pi}\int_{-N\pi}^{N\pi} f(x+\alpha/N)\frac{\sin(\alpha+\alpha/2N)}{\alpha(\sin\alpha/2N)/(\alpha/2N)}\,d\alpha \end{aligned} \tag{4.1}$$

We notice that in the numerator above

$$\lim_{N\to\infty}\sin(\alpha+\alpha/2N) = \sin\alpha \tag{4.2}$$

and in the denominator we have the limit

$$\lim_{N\to\infty}\alpha(\sin\alpha/2N)/(\alpha/2N) = \alpha \tag{4.3}$$

what is in agreement with the known limit

$$\lim_{\vartheta\to 0}\frac{\sin\vartheta}{\vartheta} = 1 \tag{4.4}$$

where $\vartheta = \alpha/2N$.

In this way we receive

$$s(x) = \lim_{N\to\infty} s_N(x) = \tag{4.5}$$
$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\frac{\sin\alpha}{\alpha}\,d\alpha = \frac{1}{\pi}f(x)\int_{-\infty}^{\infty}\frac{\sin\alpha}{\alpha}\,d\alpha =$$
$$= \frac{1}{\pi}f(x)\pi = f(x)$$

We have shown that if $f(x)$ is periodic and piecewise smooth function, then its Fourier series converges pointwise to $f(x)$ at all points of continuity of f.

To compute the Dirichlet integral

$$\int_{-\infty}^{\infty}\frac{\sin\alpha}{\alpha}\,d\alpha$$

we have used the result from Sections 5 and 6.

5. Geometric series evaluation

Here we evaluate the sum of the geometric series as in [Hir]

$$S_N = \sum_{k=1}^{N} e^{ik\varphi} \tag{5.1}$$

We write

$$S_N = e^{i\varphi} + e^{i2\varphi} + e^{i3\varphi} + \ldots + e^{iN\varphi} \tag{5.2}$$

We multiply the above equation by $e^{i\varphi}$ on both sides obtaining

$$e^{i\varphi}S_N = e^{i2\varphi} + e^{i3\varphi} + \ldots + e^{iN\varphi} + e^{i(N+1)\varphi} \tag{5.3}$$

Now we subtract equation (5.3) from equation (5.2) and we receive

$$S_N - e^{i\varphi}S_N = e^{i\varphi} - e^{i(N+1)\varphi} \tag{5.4}$$

what can be rewritten as

$$S_N(1 - e^{i\varphi}) = e^{i\varphi} - e^{i(N+1)\varphi} \tag{5.5}$$

From equation (5.5) we compute the partial sum S_N as

$$S_N = \frac{e^{i\varphi} - e^{i(N+1)\varphi}}{1 - e^{i\varphi}} = \frac{e^{i(N+1)\varphi} - e^{i\varphi}}{e^{i\varphi} - 1} = e^{i\varphi}\frac{e^{iN\varphi} - 1}{e^{i\varphi} - 1} = \tag{5.6}$$
$$= e^{i\varphi}\frac{e^{iN\varphi} - 1}{e^{i\varphi/2}(e^{i\varphi/2} - e^{-i\varphi/2})} = e^{i\varphi/2}\frac{e^{iN\varphi} - 1}{e^{i\varphi/2} - e^{-i\varphi/2}} =$$
$$= e^{i\varphi/2}e^{iN\varphi/2}\frac{e^{iN\varphi/2} - e^{-iN\varphi/2}}{e^{i\varphi/2} - e^{-i\varphi/2}} = e^{i(N\varphi+\varphi)/2}\frac{e^{iN\varphi/2} - e^{-iN\varphi/2}}{e^{i\varphi/2} - e^{-i\varphi/2}} =$$
$$= e^{i(N\varphi+\varphi)/2}\frac{\sin(N\varphi/2)}{\sin(\varphi/2)}$$

Then we can write

$$S_N = \sum_{k=1}^{N} e^{ik\varphi} = e^{i(N\varphi+\varphi)/2}\frac{\sin(N\varphi/2)}{\sin(\varphi/2)} \tag{5.7}$$

and therefrom we obtain

$$\begin{aligned} S_N = \sum_{k=1}^{N} e^{ik\varphi} &= \sum_{k=1}^{N}(\cos k\varphi + i\sin k\varphi) \\ &= e^{i(N\varphi+\varphi)/2}\frac{\sin(N\varphi/2)}{\sin(\varphi/2)} = \\ = \cos((N+1)\varphi/2)\frac{\sin N\varphi/2}{\sin\varphi/2} &+ i\sin((N+1)\varphi/2)\frac{\sin N\varphi/2}{\sin\varphi/2} \end{aligned} \tag{5.8}$$

what gives

$$\sum_{k=1}^{N}\cos k\varphi = \cos((N+1)\varphi/2)\frac{\sin N\varphi/2}{\sin\varphi/2} \tag{5.9}$$

$$\sum_{k=1}^{N}\sin k\varphi = \sin((N+1)\varphi/2)\frac{\sin N\varphi/2}{\sin\varphi/2} \tag{5.10}$$

6. Dirichlet integral evaluation

We have written previously in Section 2 that

$$\frac{\sin(N\varphi+\varphi/2)}{2\sin\varphi/2} = \frac{1}{2} + \sum_{k=1}^{N}\cos k\varphi \tag{6.1}$$

We integrate both sides of the above equation from $-\pi$ to π

$$\int_{-\pi}^{\pi}\frac{\sin(N\varphi+\varphi/2)}{2\sin\varphi/2}\,d\varphi = \frac{1}{2}\int_{-\pi}^{\pi}d\varphi + \sum_{k=1}^{N}\int_{-\pi}^{\pi}\cos k\varphi\,d\varphi \tag{6.2}$$

receiving

$$\int_{-\pi}^{\pi}\frac{\sin(N\varphi+\varphi/2)}{2\sin\varphi/2}\,d\varphi = \pi \tag{6.3}$$

because

$$\int_{-\pi}^{\pi}\cos k\varphi\,d\varphi = 0 \tag{6.4}$$

for any value of k in the sum $\sum_{k=1}^{N}$ above.

Now we substitute as before $N\varphi = \alpha$. Then we have $\varphi = \alpha/N$, $\varphi/2 = \alpha/2N$, $d\varphi = d\alpha/N$. If $\varphi = -\pi$ then

$\alpha = -N\pi$ and if $\varphi = \pi$ then $\alpha = N\pi$. After substitutions we may write that

$$\int_{-N\pi}^{N\pi} \frac{\sin(\alpha + \alpha/2N)}{2N \sin \alpha/2N}\, d\alpha = \pi \tag{6.5}$$

and after taking the limit N approaching infinity we arrive at

$$\int_{-\infty}^{\infty} \frac{\sin \alpha}{\alpha}\, d\alpha = \pi \tag{6.6}$$

in a similar way like at the evaluation of the limit of partial sum of the Fourier series.

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