Proof of the Fourier integral theorem using integral substitution with limit and the Dirichlet integral value

Pawel Jan Piskorz

Abstract

We propose a procedure which allows to prove the Fourier integral theorem using substitutions with limits in the Fourier integral and the Dirichlet integral value.

1 Introduction

In the Fourier integral in [1] on page 188 we introduce a limit and two substitutions to prove that it equals to f(x) which is a function absolutely integrable on the whole Ox axis. In our proof we will use the Dirichlet integral value which equals to π.

2 Fourier integral theorem

The Fourier integral theorem states that

$$\frac{1}{\pi}\int_0^\infty d\lambda \int_{-\infty}^\infty f(u)\cos\lambda(u-x)\,du \tag{2.1}$$

equals to the function $f(x)$.

We can prove it as follow. We suppose that $f(x)$ is absolutely integrable on the whole Ox axis. Then we can write

$$\frac{1}{\pi}\int_0^\infty d\lambda \int_{-\infty}^\infty f(u)\cos\lambda(u-x)\,du = \tag{2.2}$$

$$= \lim_{L\to\infty}\frac{1}{\pi}\int_0^L d\lambda \int_{-\infty}^\infty f(u)\cos\lambda(u-x)\,du$$

and we obtain

$$\int_0^L d\lambda \int_{-\infty}^{\infty} f(u) \cos \lambda(u-x)\, du = \tag{2.3}$$

$$= \int_{-\infty}^{\infty} du \int_0^L f(u) \cos \lambda(u-x)\, d\lambda =$$

$$= \int_{-\infty}^{\infty} f(u) \frac{\sin L(u-x)}{u-x}\, du$$

We introduce substitution $u - x = v$ and receive

$$\int_{-\infty}^{\infty} f(x+v) \frac{\sin Lv}{v}\, dv \tag{2.4}$$

Now we substitute $Lv = w$ and we have

$$\int_{-\infty}^{\infty} f(x+w/L) \frac{\sin Lv}{Lv}\, dLv = \int_{-\infty}^{\infty} f(x+w/L) \frac{\sin w}{w}\, dw \tag{2.5}$$

We can state that

$$\frac{1}{\pi} \int_0^{\infty} d\lambda \int_{-\infty}^{\infty} f(u) \cos \lambda(u-x)\, du = \tag{2.6}$$

$$= \lim_{L\to\infty} \frac{1}{\pi} \int_{-\infty}^{\infty} f(x+w/L) \frac{\sin w}{w}\, dw = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \frac{\sin w}{w}\, dw =$$

$$= \frac{1}{\pi} f(x) \int_{-\infty}^{\infty} \frac{\sin w}{w}\, dw = \frac{1}{\pi} f(x) \pi = f(x)$$

what concludes our proof of the Fourier theorem.

References

[1] G.P. Tolstov, *Fourier Series* (1976), Translated from the Russian by R.A. Silverman, Dover Publications, Inc., New York.